On the Integration of Highly Oscillatory Functions

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Abstract

The integration of rapidly oscillating functions is of interest across a variety of fields. A numerical approach is often the best method for integrating these functions[1,2]. Our research is focused on finding an efficient method of approximation. The method extracts a first order oscillation via a Taylor polynomial expansion. A comparison of errors is made between the new method and Simpson's rule.

Motivation

The particular integral of interest in this project was $\int_a^b f(x)e^{i\omega g(x)}dx$, where f(x) and g(x) are both functions and ω is a positive integer.

Uses for this integration technique arise in the study of Fourier Series whose applications are found in but not limited to:

- heat transfer of material properties
- recording music as the sum of frequencies
- revenue streams that have an annual, daily, or hourly pattern
- image processing used in MRI's[3]

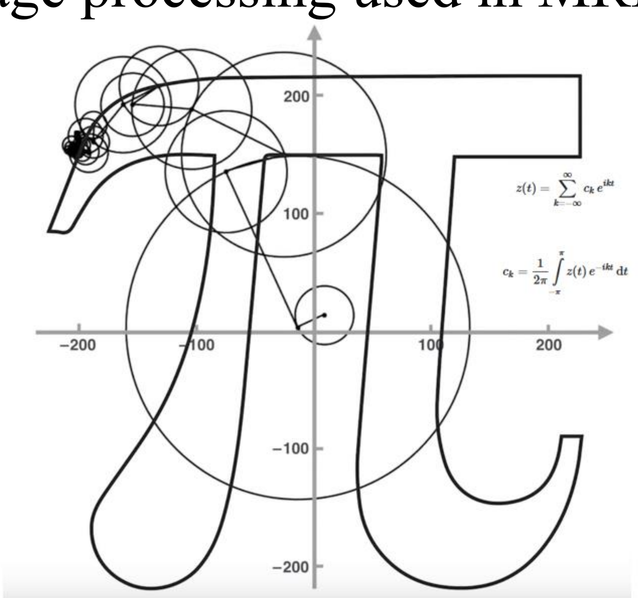


Image courtesy of Olli Niemitalo

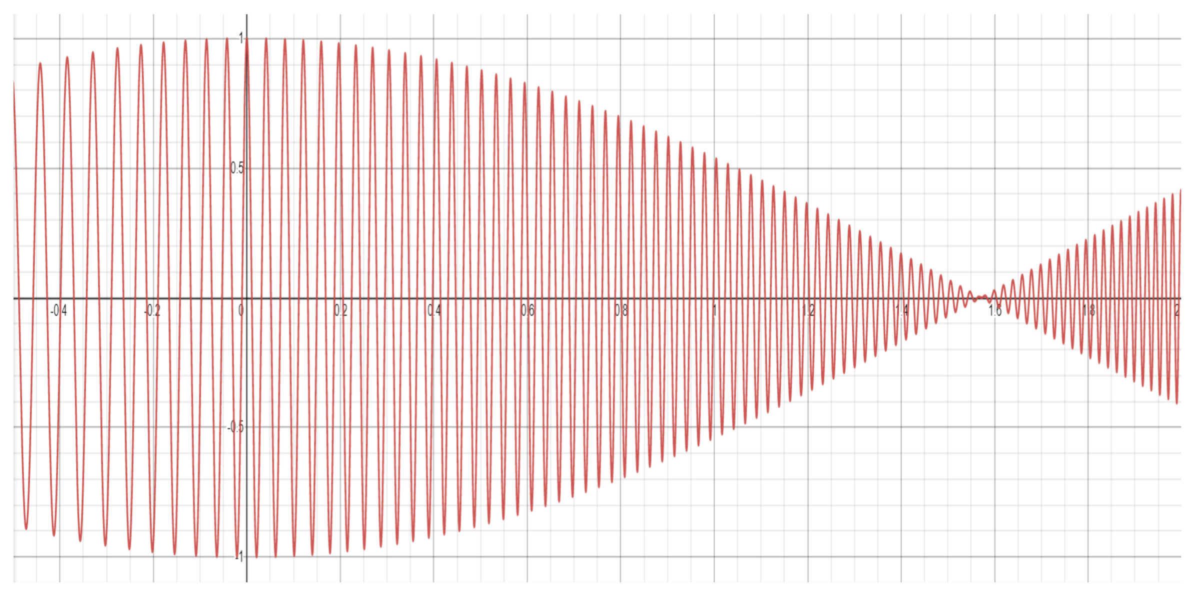
Methods

The integral $\int_a^b f(x)e^{i\omega g(x)}dx$ can be exactly integrated for a variety of functions.

These exactly integratable functions are of key importance as they allow us to easily calculate error in our own approximation as well as speed of convergence against existing methods. The test integral we chose was

$$\int_0^1 \cos(x)e^{i\omega(x^2+3x)}dx$$

Whose real part graphed on the xy-plane with ω=60 is:



The first step in our approximation is to create a first degree Taylor polynomial of g(x) under the form:

$$g(x) = g(x) + g'(m)(x-m) - g'(m)(x-m)$$

Note that m is the midpoint of the integral calculated by $\frac{a+b}{2}$

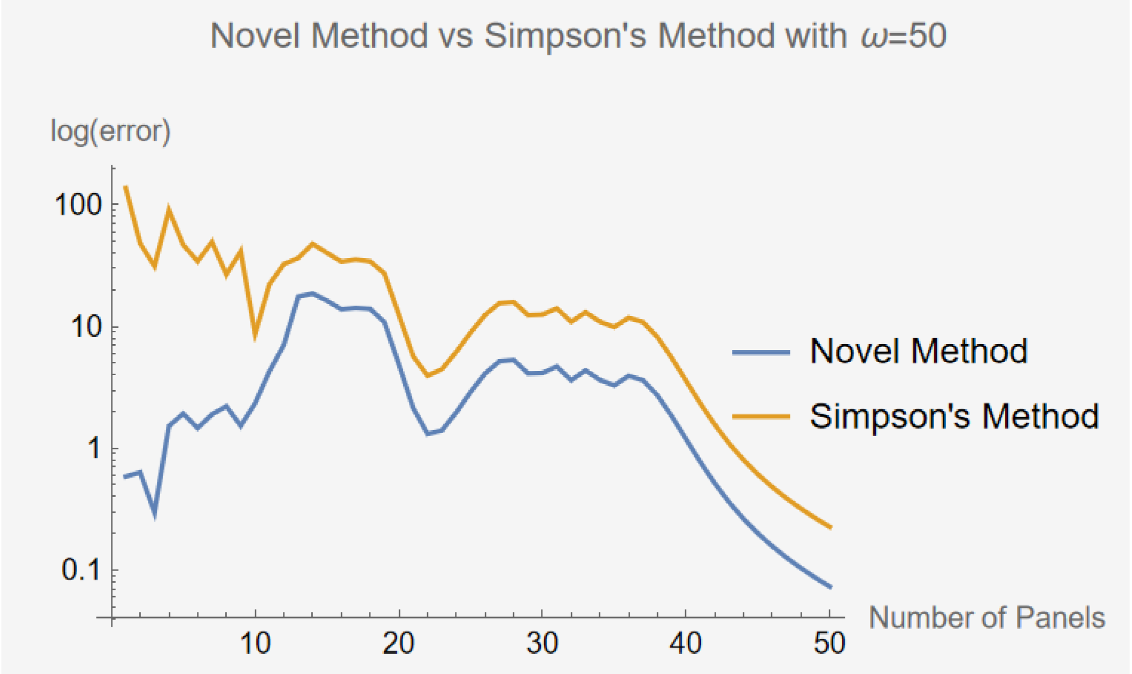
And then to approximate the f(x) term along with the $g(x) - g'(x)(x-m)$ term from the g(x) approximation to a quadratic similar to Simpson's rule, to a general form of

$$(\alpha + \beta x + \gamma x^2)e^{i\omega g'(m)(x-m)}$$

Which is a function that can always be exactly integrated.

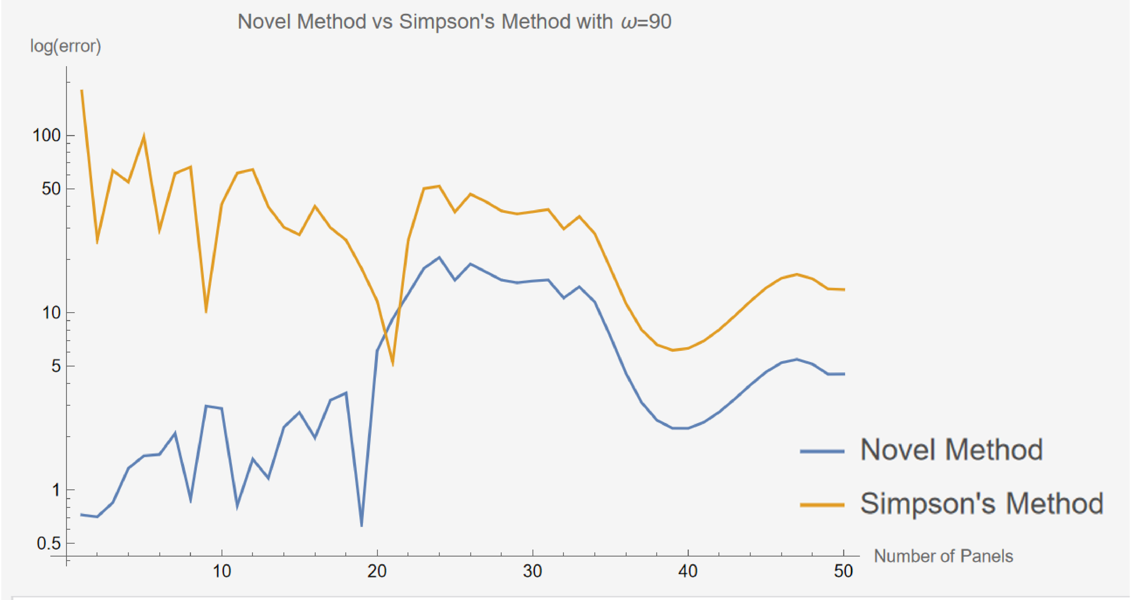
Results

The efficacy of our approximation was calculated by a comparison to Simpson's rule to the exact known value for the integral of our test function. The resulting plot shows the error on a logarithmic scale for both ours and Simpson's rule for an ω value of 50:



The error chart is mostly noise for less than approximately 50 panels. This is due to the approximating panels being used being too wide to give any meaningful results. This required number of panels before convergence depends primarily on ω.

Here is the same plot with an ω value of 90.



Conclusions

From our comparison to Simpson's rule using

$$\int_0^1 \cos(x)e^{i\omega(x^2+3x)}dx$$

it can be seen that our method is 66% more accurate than Simpson's rule for the same number of panels.

We believe that this improvement is because we maintain the oscillatory element $(e^{i\omega g'(m)(x-m)})$ in our approximation.

Future Work

Future work will be focused on two methods of improvement.

One route is to increase the accuracy of the approximation through higher order approximations for the f(x) term as well as a 2nd degree Taylor polynomial approximation for the g(x) term.

Another viable route of increasing the efficiency of our method is to streamline the algorithm that computes the integral.

Acknowledgements

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References

1) Huybrechs and Olver. *Highly oscillatory quadrature*, 2009
2) Hillion and Nurdin, *Integration of highly oscillatory functions*, 1977
3) Miller. *The Fourier Transform and MRIs*, 2012

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