change the name of the third dot and call it "with Marie" instead of keeping "Marie," and explicitly add the "with" to the green relationship. We may never forget that it is difficult to tell a story corresponding to the starting diagram. The teacher must be supple and let the children modify the diagram when they want to do it and exactly as they want to do it. They must have the possibility to adapt the diagram to the story they wish to tell, and the story to the diagram they want to keep, in such a way that they slowly reach a solution which satisfies them. If they succeed in building a coherent story — but a story which does not even look like the starter — the teacher must be able to accept it: the terminology, the convention, the game's control belong to the children.

One must, at all cost, avoid dogmatic use of the technique, for dogmatism kills the children's freedom of expression. We must use representation systems which, thanks to inner technical constraints, suggest to the child the use of a logic which the teacher has hidden in it.

Conclusion

A non-verbal auxiliary formalism can serve as guide to the child's thought. If this formalism, or representation system, is used in a non-dogmatic way, it enables the children to build a coherent story through successive adaptations that they suggest. This story can be graphically represented by the proposed formalism. In this case one should use the definitions formulated by some children and accepted by the whole class. Such a formalism is also useful because the teacher, when choosing the symbols and imposing upon them the technical constraints, can hide a logic in the system. The teacher can thus choose a logic which the children will use nearly spontaneously. Moreover, such a formalism enables the teacher to visualize the difference between object-language and metalanguage.

References

Lowenthal, F. (1980a). "Language learning and logic," in *Cognitive Development Research in Science and Mathematics,* W. F. Archenhold, R. H. Driver, A. Orton, and C. Wood-Robinson, eds., pp. 121-128, University of Leeds, Leeds.

Lowenthal, F. (1980b). Un formalisme auxiliaire: pourquoi? Comment l'utiliser? *Revue de Phonétique Appliquée, 55/56,* 263-266.

Lowenthal, F. (in press). Development of language and mathematical concepts in children: is there a relationship? in *Proceedings of the IVth International Conference for Mathematics Education* (Berkeley, 1980).

Papy, F. et G. (1968). *L'enfant et les graphes,* Didier, Bruxelles.

Communicating Mathematics: Surface Structures and Deep Structures

Richard R. Skemp

A distinction is made between the surface structures (syntax) of mathematical symbol-systems and the deep structures (semantics) of mathematical schemas. The meaning of a mathematical communication lies in the deep structures — the mathematical ideas themselves, and their relationships. But this meaning can only be transmitted and received indirectly, via the surface structures; correspondence between deep and surface structures is only partial. Some resulting problems of communicating mathematics are discussed, and some remedies suggested.

The power of mathematics in enabling us to understand, predict, and sometimes to control events in the physical world lies in its conceptual structures — in everyday language, its organised networks of ideas. These ideas are purely mental objects: invisible, inaudible, and not easily accessible even to their possessor. Before we can communicate them, ideas must become attached to symbols. These have a dual status. Symbols are mental objects, about which and with which we can think. But they can also be physical objects — marks on paper, sounds — which can be seen or heard. These serve both as labels and as handles for communicating the concepts with which they are associated. Symbols are an interface between the inner world of our thoughts, and the outer, physical world.

These symbols do not exist in isolation from each other. They have an organisation of their own, by virtue of which they become more than a set of separate symbols. They form a symbol system. A symbol system consists of

a set of symbols	corresponding to	a set of concepts

together with

a set of relations between the symbols	corresponding to	a set of relations between the concepts.

What we are trying to communicate are the conceptual structures. *How* we communicate these, or try to, is by writing or speaking symbols. The first are what is most important. These form the *deep structures* of mathematics. But only the second can be transmitted and

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received. These form the *surface structures.* Even within our minds the surface structures are much more accessible, as the term implies. And to other people they are the only ones which are accessible at all. But the surface structures and the deep structures do not necessarily correspond, and this causes problems.

Here are some examples to illustrate the differences between a surface structure and a deep structure.

I feel like a wet rag	
	→ Same surface structure, different deep structure
I feel like a glass of beer	
	→ Same surface structure, same deep structure
I feel like a cup of tea	
	→ Different surface structure, same deep structure
Shall I put the kettle on?	

What has this to do with mathematics? At a surface level wet rags and cups of tea would seem to have little connection with mathematics. But at a deeper level, this distinction between surface structures and deep structures, and the relations between them, is of great importance when we start to think about the problems of *communicating* mathematics.

For convenience let us shorten these terms to S for surface structure, D for deep structure. S is the level at which we write, talk, and even do some of our thinking. The trouble is that the structure of S may or may not correspond well with the structure of D. And to the extent that it does not, S is inhibiting D as well as supporting it.

Let us look at some mathematical examples. We remember that a symbol system consists of:

(i) a set of symbols, e.g. 1 2 3 ...
1/2 3/4 ...
a b c ...

(ii) one or more relations on those symbols, e.g. order on paper (left/right, below/above); order in time, as spoken.

But since the essential nature of a symbol is that it represents something else — in this case a mathematical concept — we must add

(iii) such that these relations between the symbols represent, in some way, relations between the concepts.

So we must now examine what ways these are, in mathematics. Here is a simple example. (Note that 'numeral' refers to a symbol, 'number' refers to a mathematical concept.)

Symbols	*Concepts*
(i) 1 2 3 ... (numerals in this order)	the natural numbers
Relations between symbols	*Relations between concepts*
(ii) is to the left of (on paper) before in time (spoken)	is less than

This is a very good correspondence. It is of a kind which mathematicians call an isomorphism. Place value provides another well known example of a symbol system.

Symbols	*Concepts*
(i) 1 2 3 ... (numerals)	natural numbers
Relations between symbols	*Relations between concepts*
(ii) numeral_1 is one place left of numeral_2.	number_1 is ten times number_2.

By itself this is also a very clear correspondence. But taken with the earlier example, we find that we now have the same relationship between symbols, *is immediately to the left of,* symbolising two different relations between the corresponding concepts: *is one less than* and *is ten times greater than.* We might take care of this at the cost of changing the symbols, or introducing new ones; e.g., commas between numerals in the first example. But what about these?

23   $2^1/_2$ 2a

These can all occur in the same mathematical utterance. And this is not just carelessness in choice of symbol systems; it is inescapable, because the available relations on paper or in speech are quite few: left/right, up/down, two dimensional arrays (e.g., matrices); big and small (e.g., 7, r) What we can devise for the surface structure of our symbol system is inevitably much more limited than the enormous number and variety of relations between the mathematical concepts, which we are trying to represent by the symbol system.

Looking more closely at place value, we find in it further subtleties. Consider symbol: 5 7 2. At the S level we have three numerals in a simple order relationship. But at the D level it represents

(i) three numbers, corresponding to 5 7 2

↕ ↕ ↕

(ii) three powers of ten: 10^2 10^1 10^0

These correspond to the three locations of the numerals, in order from right to left.

(iii) three operations of multiplication: the number 5 multiplied by the number 10^2 (= 100), the number 7 multiplied by the number 10^1 (= 10), the number 2 multiplied by the number 10^0 (= 1).

(iv) addition of these three products (5 hundreds, seven tens, two).

Of these four at D level, only the first is explicitly represented at S level by the numeral 572. The second is implied by the spatial relationships, not by any visible mark on the paper. And the third and fourth have no symbolic counterpart at all: they have to be deduced from the fact that the numeral has more than one digit.

Once one begins this kind of analysis, it becomes evident there is a huge and almost unexplored field—enough for several doctoral theses. For our present purposes, it is enough if we can agree that the surface structure (of the symbol system) and the deep structure (of the mathematical concepts) can at best correspond reasonably well, in limited areas, and for the most part correspond rather badly.

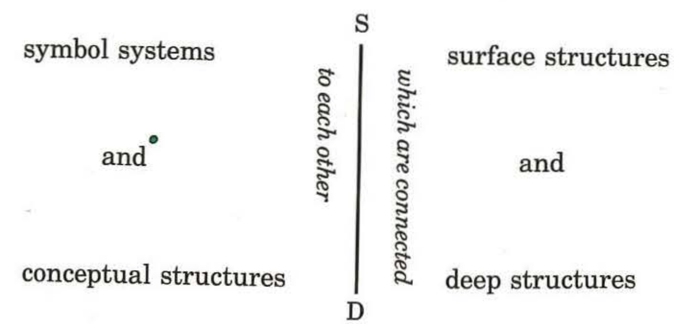
To help our thinking further in this difficult area, I would like to introduce two further ideas. The first comes from my new model of intelligence (Skemp 1979) and does not require any other parts of the theory. It is based on the well-known phenomenon of resonance. "The starting point is to suppose that conceptualised memories are stored within tuned structures, which, when caused to vibrate, give rise to complex wave patterns.... Sensory input which matches one of these wave patterns resonates with the corresponding tuned structure, or possibly several structures together, and thereby sets up the particular wave pattern of a certain concept." (page 134)

It is convenient at this stage to introduce the term *schema,* which is simply a shorter way of referring to a conceptual structure. A schema (i.e., a conceptual structure stored in memory) thus corresponds in this model to a particular tuned structure. We all have many of these tuned structures corresponding to our many available schemas, and sensory input is interpreted in terms of whichever one of these resonates with what is coming in. What is more, different structures may be thus activated by the same input in different people, and at different times in the same person. Different interpretations will then result. For example, the word

'field' will have quite different meanings according as it evokes resonances corresponding to the schemas in advanced mathematics, electromagnetism, cricket, agriculture, or general scholarship.

The second idea is due to Tall (1977) who has suggested that a schema can act as an attractor for incoming information. He took the idea from the mathematical theory of dynamic systems; but if we now combine it with the resonance model, we can offer an explanation of how this attraction might take place. Sensory input will be structured, interpreted, and understood in terms of which ever resonant structure it activates. In some case, more than one resonant structure may be activated simultaneously, and we can turn our attention at will to one or the other. In others, one schema captures all the input. (This 'capture effect' is well known to radio engineers, who have put it to good use.)

So we may now synthesise the following ideas.



Note that in the above diagram each point represents not a single concept but a schema, in the same way as a dot on an airline map can represent a whole city—London, Atlanta, Rome.

How can this theoretical model help our thinking, and what are the practical consequences? All communication, written or oral, is necessarily into the symbol system at S. *To be understood mathematically, it must be attracted to D.* This requires that D is a stronger attractor than S. If it is not, *S will capture the input,* or most of it.

One of the advantages of a good model is that it points up some questions we should ask next. The first is clearly: What are the conditions for D to be a strong attractor? Another is: can D capture the input instead of S? If so what happens?

I will take the second first, briefly. If this were to happen, I think it would mean that all the mathematical activity was confined to a deep conceptual level, and was not 'escaping' to a symbolic level at all. This may not happen completely, but some of the high-powered mathematicians who taught me at university suggest only very limited escape to S!

Returning to the first question: what are the conditions for D to be a strong attractor? S has a built in advantage: all communicated input has to go there first. And for D there is a point of no return. In the years' long learning process, if the deep conceptual structures are not formed early on, they can never develop as attractors. For too many children, D is effectively not there. And if the D structure is absent or weak, all input will be assimilated to S: the effort to find some kind of structure is strong. So S will build up at the expense of D.

But this guarantees problems, in view of the lack of internal consistency of S. This reveals a built-in advantage of D, that it *is* internally consistent. Of all subjects, mathematics is one of the most internally consistent and coherent. So if it gets well established, input to S will evoke more extensive and meaningful resonances in D than in S, and D will attract much of the input.

Doing mathematics involves the manipulation of certain mental objects, namely mathematical concepts, using symbols as combined concepts and labels. But for many children (and adults) these objects are not there. So they learn to manipulate substitute objects: empty symbols, handles without anything attached, labels without contents. This in the long run is much more difficult to do, though unfortunately in the short run it may be easier to learn. The manipulation of mathematical concepts is helped by the nature of the concepts and schemas themselves, which give a feeling of intrinsic rightness or wrongness. This arises partly from the concepts themselves, whose individual properties contribute to how we use them and fit them together. More strongly, it comes from the schemas, which determine what are permissible and non-permissible mental actions within a given mathematical context.

The problems which so many have with mathematical symbols thus arise partly from the laconic, condensed, and often implicit nature of the symbols themselves; but largely also from the absence or weakness of the deep mathematical schemas which give the symbols their meaning. Like a referred pain, the location of the trouble is not where it is experienced. The remedy likewise lies mainly elsewhere, namely in the building up of the conceptual structures.

How can we help learners to do this? This is too large a question for a single paper, but here are some suggestions as starting points.

(i) Particularly in their early years we can give children as many physical embodiments as possible of the mathematical concepts which we want to help them to construct. As examples of units, tens, and hundreds, we can use single milk straws, bundles of ten of these, and bundles of ten bundles of ten. These correspond much more closely to the relevant mathematical concepts than do the associated symbols, and so the visual input will be attracted more strongly to the relevant parts of D than to S. In such cases, moreover, the input goes first to D, then to S, since the children are first presented with the physical embodiments of the concepts, and thereafter are asked to connect these with appropriate symbols.

(ii) By careful analysis of the mathematical structure to be acquired, we can sequence the presentation of new material in such a way that it can always be assimilated to a conceptual structure, and not just memorised in terms of symbolic manipulations. Many existing texts show no evidence that this has been done. (See Skemp 1971, Chapter 2.)

(iii) Again in these important early years, it helps children if we stay longer with spoken language. The connection between thought and spoken words are initially much stronger than those between thoughts and written words or symbols. Spoken words are also much quicker and easier to produce. So in the early years of learning mathematics, we may need to resist pressures for children to have 'something to show' in the form of pages of written work.

(iv) It is often helpful to use informal, transitional notations as bridges to the formal, highly condensed notations of traditional mathematics. By allowing children to express their thoughts in their own ways to begin with, we are using symbols which are already well attached to their associated concepts. These ways of expression may often be lengthy, unclear, and differ between individuals. By experience of these disadvantages, and by discussion, children may gradually be led to the use of established mathematical symbolism in such a way that they experience its convenience and power for communicating and manipulating mathematical ideas.

References

Skemp, R. R. (1971). *The Psychology of Learning Mathematics.* Penguin, Harmondsworth.

Skemp, R. R. (1979). *Intelligence, Learning, and Action.* Wiley, Chichester.

Tall, D. O. (1977). "Conflicts and Catastrophes in the Learning of Mathematics," *Mathematical Education for Teaching.*

Mathematical Symbolism

Derek Woodrow

One of the essential distinguishing features of mathematics is its eventual dependence upon symbols and symbolic expression. Few attempts to determine those processes, activities, or contents which uniquely identify mathematics have succeeded. It is indeed questionable whether human knowledge can be classified into such self-contained categories. The many diverse activities of mathematicians do, however, have symbolic expression as their common feature, and the extent to which modern disciplines depend upon mathematics could be measured by their growing reliance on symbols. It is reasonable to surmise that much of the difficulty experienced by children in mathematics, and the lack of popularity of physical as opposed to biological sciences in higher education, could be traced to the problem of symbolisation. It will be interesting to watch the effect on, say, geography as the school syllabuses move towards mathematical as opposed to descriptive aspects. There is surprisingly little apparent research into the use and learning of symbols, except for the many investigations into both the problem of how children learn to read and adult perceptual experiences with words (e.g., Coltheart 1972). There is, however, a real distinction between the use of symbols as a verbal language (spoken or written) and the use of symbols in the mathematical sense. It will indeed be suggested below that one activity interferes with the other.

In normal reading activity the written word contains very many redundancies. There is clear experimental evidence that not only are many of the words used unnecessary and the number of letters per word quite extravagant but the letter symbols themselves are only partially scanned in many reading techniques. The reader only notices, say, the bottom of the letter and the relationship between the symbols is sufficient to determine them completely. Try reading the following doggerel:

THR NC WS YNG MN, WHS FC WS GRN?
T WS TR THT LL WH SW HM FND HM TH STRNGST THNG
THD SN

The relationships between verbal symbols can also be seen in the way in which adults react and remember random letters. A collection of letters such as POSTIC is much more easily read and remem-

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