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Mathematical Symbolism

Derek Woodrow

One of the essential distinguishing features of mathematics is its eventual dependence upon symbols and symbolic expression. Few attempts to determine those processes, activities, or contents which uniquely identify mathematics have succeeded. It is indeed questionable whether human knowledge can be classified into such self-contained categories. The many diverse activities of mathematicians do, however, have symbolic expression as their common feature, and the extent to which modern disciplines depend upon mathematics could be measured by their growing reliance on symbols. It is reasonable to surmise that much of the difficulty experienced by children in mathematics, and the lack of popularity of physical as opposed to biological sciences in higher education, could be traced to the problem of symbolisation. It will be interesting to watch the effect on, say, geography as the school syllabuses move towards mathematical as opposed to descriptive aspects. There is surprisingly little apparent research into the use and learning of symbols, except for the many investigations into both the problem of how children learn to read and adult perceptual experiences with words (e.g., Coltheart 1972). There is, however, a real distinction between the use of symbols as a verbal language (spoken or written) and the use of symbols in the mathematical sense. It will indeed be suggested below that one activity interferes with the other.

In normal reading activity the written word contains very many redundancies. There is clear experimental evidence that not only are many of the words used unnecessary and the number of letters per word quite extravagant but the letter symbols themselves are only partially scanned in many reading techniques. The reader only notices, say, the bottom of the letter and the relationship between the symbols is sufficient to determine them completely. Try reading the following doggerel:

THR NC WS YNG MN, WHS FC WS GRN?
T WS TR THT LL WH SW HM FND HM TH STRNGST THNG
THD SN

The relationships between verbal symbols can also be seen in the way in which adults react and remember random letters. A collection of letters such as POSTIC is much more easily read and remem-

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bered than XZBQT which proves much more difficult because it does not resemble the normal letter associations used in the English language.

The redundancy which is normal in language is not usually present in mathematical symbolism at school level. Statements such as:

$$3 + 4 + 10 + 3 \cdot 2 = 20 \cdot 2$$
$$A \cap B' = \emptyset$$
$$(3,4) + (4,5) = (7,9)$$
$$4x^2 + 3x + 2 = 0$$

contains little redundancy, although the last example *with experience* can be seen to have a recognisable form in which one might only need to know the coefficients 4, 3, and 2. Even in this case, however, the relevant distinctive information is contained *inside* the symbolisation which must therefore be read rather than just seen. Yet another complication in mathematical symbolism is the phenomenon of temporary redundancy in which a whole group of symbols are at one stage carried without reading, only to need detailed reading later. For example:

$$(12x^2 - 2)^2 - 2(12x^2 - 2) + 1 = 0 \qquad 12x^2 = 3$$
$$[(12x^2 - 2) - 1]^2 = 0 \qquad x^2 = \tfrac{1}{4}$$
$$12x^2 - 2 = 1 \qquad x = \pm\tfrac{1}{2}$$

This becomes more apparent in the later stages of learning mathematics, and this variation in the degree of redundancy causes many problems for college and university students.

Another distinction between the use of words and mathematical symbols is the independence of one symbol from the preceeding and succeeding symbols. The anecdote is related of the three-year-old who was arithmetically very advanced in that the addition of three digit numbers presented little difficulty. His parents expressed some concern that he had no interest in reading; reputedly because letters behave irrationally, in the sense that whilst any sequence of digits make a sensible number a random sequence of letters do not make a word. In reading, the individual symbols do not themselves contain any meaning, whereas in mathematics, with a few exceptions such as d/dy or ( ), the meaning of the individual symbols is vital.

Even more disturbing to the learner is the interrelationship of mathematical symbols where not only does each symbol have its own distinctive meaning, but this meaning is affected by its neighbouring symbols. Consider, for example, the schema attached to the symbol 2 in 212, ½, $\sqrt{2}$, $f(2)$, a_2, a^2, $\mathbb{R}^2$, 2'o-clock, 1001_2, (2,3), etc. In each case there are subtle changes in a basic schema which originally starts as a fairly low-level concept in 2 as used in the infant natural number

sequence but becomes a higher and higher level schema as mathematics progresses.

The essential concentration in school curricula on literacy tends to produce, therefore, a reading technique which to some extent interferes with the technique required in reading mathematical symbols. If one accepts this proposition, then two implications arise: we must adapt mathematical symbolism for the learner, and we must follow a careful and structured plan to teach the pupil how to read mathematics.

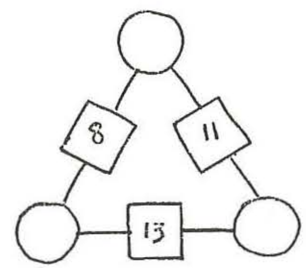

Figure 1. The arithmogon.

Signs

One of the usual ways of adapting mathematical symbols to the schema used by pupils is the use of signs such as boxes instead of symbols. Many books now in use make extensive use of boxes from a very early stage, frequently asking questions such as 3 + □ = 7. Another interesting example is the arithmogon (McIntosh & Quadling 1975). In the arithmogon (Figure 1) the numbers which belong in the square boxes are the result of adding the numbers in its adjacent circle boxes. Many points of interest arise from the investigation of what numbers should be in the circles for given numbers in the squares. What is relevant to the present argument is the difference in schema attached to this problem compared to its presentation in the usual mathematical notation. Whilst many primary children could tackle the sign statement of the problem, it is doubtful if many early secondary school pupils would be able to manage the symbolic statement.

$$\begin{cases} x+y=8 \\ x+z=11 \\ z+y=13 \end{cases}$$

Comparison of the two expressions 3 + □ = 5 and 3 + x = 5 illustrates some of this difference between signs and symbols. The first uses the sign □ to replace the missing number and the second uses x

in apparently the same way. The second expression carries with it, however, a much more abstract statement which says 'this particular example belongs to a whole class of things which can be dealt with in such-and-such a way.' In solving the first problem it really is the number which should be in the box which is the relevant factor, in the second it is the process of obtaining whatever number turns up which is relevant. Whilst this appears to be a post-operandi argument and to have import only at later stages in the learning of mathematics, experience points to the operation of such distinctions at an almost instinctive level. Children who cannot be at all aware of this distinction from experience react so differently to the use of a sign □ than a symbol x.

Rather surprisingly this distinction in the order of concept involved is echoed in adult perception. Coltheart (1972) reports an experiment in which observers are presented with a 3×4 matrix of letters or shapes. After the display has been removed the observer is asked to remember a particular subset of the display chosen on the basis of position, colour, shape, or size. In the particular problem investigated this showed the existence of a short term memory of greater detail and scope than normal recall. What was rather surprising was that this short term intensive memory apparently failed to operate as effectively when the display was a mixture of letters and digits and this distinction was used as a discriminant. This would suggest that the ability to distinguish between letters and digits is in some respect different from discrimination in position, size, shape, or colour. This might indicate, incidently, another of the great advantages of arabic place-value notation based upon position rather than earlier hieroglyphic representations which depend upon a higher level of symbol discrimination. It would be interesting to repeat the recall experiments with young children to investigate if there is any particular age at which the distinction between signs and symbols, as defined here, becomes relevant. It seems very likely that the use made in mathematics of letters for numbers is probably neither accidental nor irrelevant.

It is clear that adults do not, indeed, experience much difficulty in handling signs in normal everyday life. There has always been an immediacy and ease in the use of signs for religious, political, and social reasons. Mere reference to scarab-beetles, fish, crosses, eagles, hammers and sickles, white feathers, tudor roses, fleur-de-lyse, and so on, produce immediate images and attract schemas from our memories which are full of vividness. Freud, and modern advertisers, have made this fully conscious. Traffic signs, laundry signs, and the markings on

electronic equipment illustrate the steady growth in the use of signs in modern life. The contrast of these signs with mathematical symbols illustrates the distinctive features of a sign, which is essentially a low level naming concept which identifies a single, identifiable, non-adaptable idea. Symbols, on the other hand, are identified with high-level schemas rather than concepts, and as such are more responsive to adaptations and multiple relationship. Three different types of symbolisation have therefore been identified:

Language symbols.Contain high redundancy, great interdependence, and no individual meaning.

Signs.Contain little redundancy, not interdependent and unaffected by neighbouring signs, represent single (naming) concepts.

Symbols.Contain little redundancy, interdependent and adaptable to neighbouring symbolisms, related to schema.

The Functioning of Symbols

Skemp (1971) suggests ten different ways in which symbols are used: i Communication, ii Recording, iii Forming new concepts, iv Aiding multiple classification, v Explanation, vi Aiding reflective mental activity, vii Exhibiting structure, viii Automating routine manipulations, ix Recovering information, and x Producing creative mental activity. Not all of these are, of course, independent and more than one mode of functioning is often at play at the same time. In Skemp's clear descriptions of these roles for symbolisation certain underlying problems and ideas can be seen. At a high level of mathematics there is a clear contradiction between two characteristics of symbolic representation; the condensation which symbols achieve contrasts with their use as a precise language. Both these aspects relate to the early learning of mathematics in which symbols are used to name concepts and schemas, and yet in different contexts we change and adapt these schemas to meet different needs, without always changing the symbol. (Perhaps we need vari-focal symbols to complement the idea of vari-focal concepts presented in Skemp 1979.)

Symbols as Names

Skemp comments 'It is largely by the use of symbols that we achieve voluntary control over our thoughts,' and the ability to name a thing has always conveyed controlling power in both Greek and Nordic mythology. In answer to 'What is the largest number,' the word 'infinity' settles all discussion, and the fact that the solutions to $x^2 - x + 1 = 0$ are complex satisfies most enquiries even though the hearer may have no clearly defined meaning for the words. Mathematics is usually concerned with higher order concepts for which the defining examples are other concepts, and these can only be expressed in verbal or symbolic form. Just as the young child must have the certainty of conservation of his physical observations before being able to operate with them, so the student of mathematics must be assured of the certainty of the lower level concepts before he can build with them. One of the major roles of symbols lies in communicating and holding these concepts with others, or with oneself in internal reflection and argument. In identifying three types of listener — the 'don't knows,' the 'want to know more,' and the 'critics' — Skemp illustrates the different contexts in which schema, and hence symbols, undergo subtle changes. These range from naming a general target area in which the concept is allowed to be fuzzy but the direction of clarification is hopefully indicated, through periods in which some concepts are clarified whilst others are left vague, until in the critical stage every symbol has its own specific and precisely defined meaning. The student not only passes through these stages in turn, but passes through them more than once as concepts are continually redefined. This is not only true of high level concepts such as integration, but even early in the secondary school level the uses of π and $\sqrt{2}$ illustrate the variety of conceptual contexts. Similarly the continual redefinition of multiplication has led to the introduction of the idea of group properties in an attempt to establish a conserved concept which is unvarying enough to be built upon. In the same way the idea of function compared to relation reflects a need to distinguish between two different uses of variables which otherwise cause a disturbing vagueness.

If communication is to be meaningful, it is clear that the symbol used to signal a schema in one person must signal the same schema in his correspondent. One of the problems in the use of symbols by pupils is that the teacher has frequently condensed his early use of multiple concepts and symbols into a single one. Thus, for example, the development of the concept of subtraction involves a variety of different lower level concepts such as 'take away,' 'how much bigger,' 'what is the difference between,' out of which is generated an underlying idea. Until the child has developed this underlying concept the use of the same symbol for different concepts can cause problems, and it is important that the symbolism should mirror the different activities. On occasions it is therefore necessary to use two or three symbols (or rather signs) in the early stages. (The reader is invited to describe the activities symbolised in the following list.)

$5-3=2$

$(5,3)\Rightarrow 2$

$5+\square=2$

$5\xrightarrow{-3}2$

$5+{}^{-}3=2$

$\longrightarrow 5-3=2$

The same tendency for symbols to outreach the attained concept can be seen in the use of ${}^{-}1$ and ${}^{+}3$ for the directed numbers, and many teachers have encountered difficulty which pupils have with notations such as $(\ )^{-1}$. This really denotes a multiplicative inverse, but in situations such as $\sin^{-1}x$ the connection with any multiplication situation is far from obvious. Even the equivalence of $\frac{3}{7}$ and $3 \div 7$ is not at all easy to establish.

The Introduction of Symbols

There is an apparent confusion in the work of both Skemp and Dienes on the introduction of the name of a concept (its symbol) early in the learning process. To quote from Dienes (1964): 'The most likely reason for the general ossification of mathematics in children's minds at an early stage is the rash use of symbols, i.e., the introduction and manipulation of symbols before adequate experience has been enjoyed of that which is symbolised' and 'the apprehension of structures and the symbolisation process are not altogether distinct, and in fact there is reason to believe that each acts as a stimulus on the other.'

Similarly Skemp (1971): 'Making an idea conscious seems to be closely connected with associating it with a symbol' and 'Concepts of a higher order than those which a person already has cannot be communicated to him by definition, but only by arranging for him to encounter a suitable collection of examples.'

Both writers are really talking about symbols as representing structures, (central unifying ideas, schemas) as compared to signs representing low level concepts. Skemp makes the point that there is a distinction between reflecting on content and reflecting on form which is relevant in this context, since the level of content is that of naming concepts. This distinction clearly relates to Piaget's distinction between concrete operational and formal operational thinking. The usually suggested ages for maturing from one mode to the other (in general between about 12 and 16 years of age) would indicate a need to persist with less flexible signs related to content rather than symbols related to form. The timing of this change from the particular to the abstract is implicit in the good teacher, but there has been little research to make it explicit and therefore more transferrable. (The Concepts in Secondary Mathematics Project reported by Hart [1981] has produced some interesting work in this area.)

The premature introduction of symbols to represent structures leads to pupils developing incorrect and inflexible schemas. Once a schema is established it tends to be firmly held, and pupils tend to alter their perception of contradicting concepts in order to fit them into their schema. One difficulty which many secondary pupils have in reading problems is that they construe the words so as to fit their firm schema rather than accept the intended meaning. This is one of the problems with the traditional model example and practice and theorem followed by rider methods of teaching mathematics which leads to externally imposed schema at too early a stage. This tends to encourage inflexibility and hence later a limited range of application of the schema. The method does, however, give those gifted pupils who can accommodate and change their schema an appreciation of the structure and a language in which to discern the form of mathematics.

The recent trend towards individualised learning methods, on the other hand, do give the pupils a broad base of low level concepts from which schema can be built. They allow the pupil to mark out the territory of a symbol by using it initially more as a temporary sign for a limited content, related to a short piece of work. These methods, however, seem to have difficulty in developing symbols relating to underlying structures. Because the pupils are using low level signs, they are not easily led to consider high level relationships. This absence of a symbolic language in which to recognise higher concepts leads the pupil to concentrate on easier low level concepts for which the language is available. The broad base of the triangle of mathematical knowledge which these methods create can be dissipated unless the pupils are also given the language and encouragement to build from this base.

The changes in content during the 1960's led to a considerable increase in the use of symbols; the introduction of set notation, functional notation, vectors, matrices, symbols for inverses, magnitudes, and logic. That this plethora of symbols did not cause any real disturbance might superficially seem a little surprising. The introduction of extra symbolism, however, serves to give the pupil more language in which to express and refine his ideas. Many of these new symbols were also operating at the level of signs, representing low level concepts

and distinguishing between ideas which were otherwise confused within the same symbol. One of the major problems which did become apparent was the insecurity of teachers with this symbolism, and this led to a pedantry which was out of step with the initial intentions of some reformers but which nevertheless came to be one of the characteristics of the changes. When a symbol is not securely understood, the edges of its meaning are avoided. There was a confusion, too, between the use of a symbol in the classroom for a single concept and the use of the symbol in more developed mathematics for a whole structural idea. This was enhanced by the teacher's own enjoyment in having mastered something new and wishing to pass onto the pupil immediately this whole concept of mathematical structure which had often escaped him (the teacher) in the past. The halo effect caused by this is unfortunately transitory. To mix the metaphors thoroughly, jumping from bandwagon to bandwagon can be exhilarating for teacher and pupil but is ultimately very tiring!

Whilst some features of these reforms will gradually disappear, some of the notational innovations will continue to prove advantageous. The contrast of the algebra of vectors and matrices with the algebra of number serves to help identify the more usual manipulations and encourage an appreciation of their structure. The availability of a symbol for magnitude can serve usefully to identify this particular idea from within more complex concepts (provided that it is used when required and not when it is superfluous). The idea of placeholders, solution sets, and function have not so far proved effective in the crucial problem of dealing with variables. The variety of concepts attached to, say, $y = 3x + 2$ needs a much more varied notation in the early stages. The confusion between when x and y are specific values (e.g., simultaneous equations) or representational values (drawing graphs or expressing a generalisation) or true variables (expressing abstract conceptual relationships) is present throughout mathematics and only sophisticated schema can really distinguish between them and accept their equivalencies. Many situations we present to students contain all three meanings at different stages within the same problem and the students certainly have difficulty and uncertainties as a result. The teacher, indeed, has subsumed these concepts into one schema needing one symbol, and since he does not need to differentiate he loses the facility.

The introduction of the ideas of functions and relations for use in different situations was an attempt to clarify this for the pupil, but the discrimination is only partly accomplished, and the general tendency to adopt only one or the other notation regardless of the problem concerned led to little overall improvement. More flexibility is certainly needed in the early stages of algebra, and less pedantry. Use could be made of boxes, circles, triangles, etc., when specific values are intended, and pupils should be encouraged to invent symbolism for unknown quantities and for representational situations such as generalised statements for patterns, e.g., sequences. The pupil's recognition of a need is often the best springboard for symbolism, and that symbolism must reflect that need. Mathematicians, indeed, use a great deal of implicit discriminants, such as using x, y, z for variables, a, b, c for coefficients, k, l, m for constants, and even using Greek letters, 'curly' letters, and so on. These distinctions are not readily discernable by the learner nor always conscious in the teacher and more distinct symbolisation is needed in the early stages, with the more usual conventions being allowed to grow slowly.

Symbols which Unify and Separate

One of the recurring problems is the use of a symbol on one hand to distinguish between concepts and on the other to unify concepts into more useful general schema which ignore irrelevancies. The result is that pupils cannot focus on either the woods or the trees. The value of symbols in developing simplifying structural schemas is very evident. The idea of differentiation as an operator leads to $(D^2 + 2D + 1)y = 0$ with an immediate recall of a known schema, and the possibility of extension to higher order. At a similar level of study the introduction of complex numbers in the form $re^{i\theta}$ can, and should, be dramatic. Indeed the variety of forms of complex numbers is also a good example of the use of symbols to distinguish between different facets of the same concept. Similarly at an earlier stage the use of different expressions— $16 = 2 \times 8 = 7 + 9 = 4^2 = 5^2 - 3^2 = \ldots\ldots$ —serve to emphasise different features. The expressions $16 = 1 + 15 = 2 + 14 = 3 + 13 = \ldots\ldots$ identify both different partitions and also a common feature. In introducing set language the need for a set to be well-defined is usually stressed, followed very soon by Venn diagrams in which the only specification is 'subsets of the Universal set.' Particularise, for example, the situations shown in Figure 2.

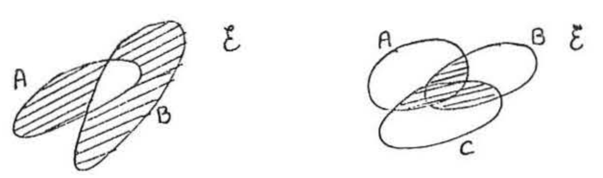

Figure 2.

The use of $a^{1/2}$ is an interesting situation in that it is at the same time the 'opposite' or 'inverse' of a^2 and also merely an extension of the exponential process $a^3\ a^2\ a^1 \ldots a^0 \ldots$. Consider the distinctions and similarities of the three statements:

A. $x + y = 7$, $3x + 2y = 15$

B. The lines $x + y = 7$ and $3x + 2y = 15$ and their intersection.

C. $\begin{pmatrix}1 & 1\\ 3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}7\\ 15\end{pmatrix}$

In A, x and y are specific particular values whilst in B there are two different independent variables and two different dependent variables which (since they are in one plane) can take the same values simultaneously at the intersection. In C, x and y are characteristics of a single vector quantity. For the teacher with a well developed schema of ordered pairs there is value in using the same letters in all three situations. The pupil is likely to be in a situation similar to the pre-conservation era of Piagetian theory and unable to appreciate the constancies within these three examples. They therefore serve merely to confuse until 'the penny drops.' Unless the appreciation occurs reasonably early in the learning process this confusion passes past its initial usefulness (in altering the pupils' schemas into more useful ones based on higher level categorisations) into dismay and rejection.

The linear function $f(x) = mx + c$ (or $y = mx + c$) is another interesting example. To the teacher, conscious of many other functions, the role of m and c in determining the behaviour of the function is very clear. To the pupil this is hardly a linear function at all but many different functions, since the importance of linearity only arises in contrast to many non-linear functions. His concentration is solely on the many values of m and c and therefore each function is distinct and individual.

Nevertheless, without the use of similar notations the crucial structural categorisations may remain hidden. What is needed at school level are notations in which both similarities and differences are evident. At a higher level such notations are normal, for example d/dx and $\partial/\partial x$, $\int$ and $\oint$, log and $\ln x$, $\sin x$ and $\sinh x$. The need of mathematicians for this kind of clued notation has not been reflected in our school notations where the need is likely to be much greater.

Some Tentative Implications

The attachment of symbols to structural schemas rather than simple concepts would suggest that they come into play only in the latter stages of learning mathematics. This is related to the teaching feature stressed by Skemp in the use of one sign or symbol for one concept or schema *of the learner*. In the early learning of algebra, symbols are used not only for different concepts but also for different types of concept such as particulars (missing numbers), generalisations (extensions of pattern), and abstractions (expressions for structure or form). These are distinctive elements, which being distinctive need distinctive notations.

There has been a tendency for some years to use non-literal signs such as boxes in the early stages of the algebra of unknowns, and the introduction of the term 'placeholder' was indicative of this trend. In the early 1960's R. G. Davies made an interesting film of a lesson in which one group of children devise a relation between △ and □ to produce an answer ○. The other children by specifying trial numbers for △ and □ and being told the resultant ○ try to establish the relation. These introductions to algebra have never, however, succeeded in becoming more than trends. The arguments in this article suggest a much greater extension and development of their use. The introduction of literal signs brings with it a greater feeling of permanence, and it is essential that this permanence does not also produce rigidity, since even simple concepts must adapt and change as maturity and sophistication grow. The discussion has led to a plea for a greater range and variety of literal symbols in the early stages, which can both serve to distinguish and unify. The arbitrary and indiscriminate use of any letter in addition to the ubiquitous x does not in itself satisfy both these requirements, but a carefully thought-out development in which similar situations had similar but distinct notations is needed. One common example is the use of bold or italic letters for vectors, points, and magnitudes. In establishing the underlying structures of which algebra is the manifestation not nearly enough attention has usually been paid to the importance of having non-examples available to help establish characteristic qualities. In particular, the concentration on an analysis of linear functions in most school syllabuses is attempted without sufficient attention to establishing the concept of linear functions. Indeed, the idea of operators and function machines rather than more general functions would seem much more pertinent in school mathematics, since the pupil has a much greater variety of experience upon which to draw. This is also reflected, perhaps, in the complaints of teachers of other subjects to whom the higher level idea of a function seems hardly as relevant. They desire the ability to manipulate single operators in sequence, whereas the concept of a function is an appreciation of the results of combining multiple operations.

This approach leads to a stress early in the course on topics similar to the traditional transformation of formulae but placed in a less algebraic setting by the use of diagrams and flow charts; e.g., such

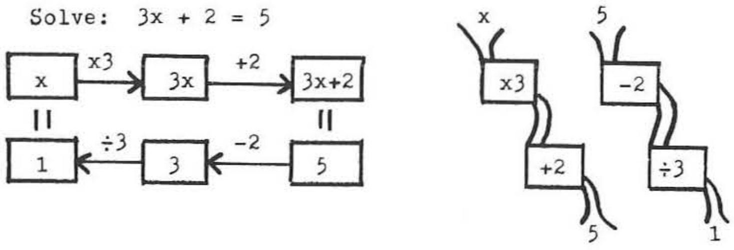

Figure 3.

representation as shown in Figure 3. This approach could build upon some of the work contained in some primary school syllabuses—(see Fletcher 1971). This use of operators leads naturally to the use of functional notation for combinations of operators at a later school stage. In the early stages of studying functions, pedantic mathematical distinctions of notation should not be demanded of the pupils, even though the teacher may well choose his own notation for a situation in anticipation of more advanced criteria. As the pupils' schemas develop so the notation can be refined; all that is necessary is the availability of suitable notations when the need for these refinements arise.

It is likely that we teachers will find it difficult to alter our own notational schemas to fit the pupils' needs. Just as teachers deplore the inability of their pupils to solve $a + bx + cx^2 = 0$ so we may have difficulty in making such simple but useful adaptations as using $Ax^2 + Bx + C = 0$ or $f(x) = Mx + C$. Such a change may seem trivial to the teacher who intuitively distinguishes between coefficients and variables (unknowns?). The change in size of notations, however, suggest such a distinction much more clearly to the pupil.

The importance of symbolism in mathematics is indisputable, but we have little research evidence on the learning of mathematical symbols. There is a great deal of expertise known to experienced teachers. Much is accomplished by hand-waving and individual ad-hoc symbols, but this needs to be externalised and theorised so as to become available to the whole community of mathematics teachers and to help overcome deficiencies of both syllabuses and texts. The lack of teachers with a secure and sound training in mathematics is unlikely to be overcome quickly, and without security there is no flexibility. It is therefore increasingly urgent that advice which rests upon a systematic and realistic theory of learning is made available.

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